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A note on the application of non-extensive statistical mechanics to fully developed turbulence

Bhimsen K Shivamoggi[1] **and Christian Beck**[2]

Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, USA

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Abstract

We show that the recent application of non-extensive statistical mechanics to fully developed turbulence (FDT) satisfies Novikov's inequality only in the near-Gaussian limit and exhibits compatibility with various other formulations of FDT. We define relations between the non-extensivity parameter q and the corresponding parameters of the lognormal, multifractal and random-β model.

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1. Introduction

Tsallis and co-workers [1, 2] have recently extended Boltzmann–Gibbs thermodynamics by generalizing the concept of entropy to the non-extensive regime. This approach has turned out to be fruitful in dealing with the statistical and multi-fractal properties of systems at critical points with long-range interactions (see Tsallis [3] for an extensive bibliography on this). Recently, one of us [4] developed an analysis of fully developed turbulence (FDT) based on the assumption that the underlying statistics of the system follows Tsallis' non-extensive prescription. In this paper, we wish to show that the Tsallis model (TM) of [4] satisfies Novikov's [5] inequality only in the near-Gaussian limit, and is compatible with the multi-fractal formulation (Meneveau and Sreenivasan [6]) and the probabilistic-cascade (the so-called random-β) model (Benzi *et al* [7]) in a modified form (Shivamoggi [8]). Furthermore, the TM is shown to be compatible with the log-normal model in the near-Gaussian limit.

2. Tsallis non-extensive model of FDT

The TM introduced in [4] uses for the radial velocity difference u between two points in the fluid separated by a distance r the probability distribution of a canonical ensemble in Tsallis' [2] non-extensive statistical mechanics. This is given by

$$p(u) = \frac{1}{Z_q}\left[1 + \tfrac{1}{2}(q-1)\beta u^2\right]^{1/(1-q)} \tag{1a}$$

[1] Permanent address: University of Central Florida, Orlando, FL 32816-1364, USA.
[2] Permanent address: Queen Mary College, University of London, London E1 4NS, UK.

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where Z_q is the partition function

$$Z_q \equiv \int_{-\infty}^{\infty} \left[1 + \tfrac{1}{2}(q-1)\beta u^2\right]^{1/(1-q)} \mathrm{d}u \tag{1b}$$

and q is the non-extensitivity parameter. One has $1 \leqslant q < 3$ in order to ensure the existence of the above integral. In the limit $q \Rightarrow 1$, equation (1) reduces to the familiar Boltzmann–Gibbs expressions:

$$p(u) = \frac{1}{Z_q}\mathrm{e}^{-\frac{1}{2}\beta u^2} \tag{2a}$$

where

$$Z_q = \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\beta u^2}\,\mathrm{d}u = \sqrt{\frac{2\pi}{\beta}}. \tag{2b}$$

If the structure function of order m scales in an inertial range according to

$$\langle |u|^m \rangle \sim r^{\zeta_m} \qquad L \gg r \gg \ell_d \tag{3}$$

where L is a large scale and ℓ_d is a dissipation scale, then, using (1) and simple model assumptions, in [4] the following formula was obtained:

$$\zeta_m = \frac{m}{3} + \log_2\left[\left\{1 - m\left(\frac{q-1}{3-q}\right)\right\}\left\{1 - 3\left(\frac{q-1}{3-q}\right)\right\}^{-m/3}\right] \qquad 1 \leqslant q < 3. \tag{4}$$

In the limit $q \Rightarrow 1$, equation (4) reduces to the familiar Kolmogorov [9] result,

$$\zeta_m = \frac{m}{3}. \tag{5}$$

The most important difference between a TM of FDT and other models of FDT is that only finitely many moments $\langle |u|^m \rangle$ exist. Since for large u one has $p(u) \sim u^{2/(1-q)}$, this implies $u^m p(u) \sim u^{m+2/(1-q)}$, and existence of the mth moment thus requires

$$m + \frac{2}{1-q} < -1$$

or

$$m < \frac{3-q}{q-1}. \tag{6}$$

Since in practice q is close to l, the number m can be rather large.

3. Comparison with the log-normal model

For $q \approx 1$, which corresponds to the near-Gaussian limit, equation (4) gives

$$\zeta_m \approx \frac{m}{3} + \frac{m(3-m)}{8\ln 2}(q-1)^2 \qquad q \approx 1. \tag{7}$$

The log-normal model (Monin and Yaglom [10]) gives, on the other hand, for the scaling exponent

$$\zeta_m = \frac{m}{3} + \frac{\mu m}{18}(3-m) \tag{8}$$

where μ is the scaling exponent of the energy dissipation (denoted by ε) correlation function,

$$\langle \varepsilon(\boldsymbol{x})\varepsilon(\boldsymbol{x}+\boldsymbol{r}) \rangle \sim r^{-m}. \tag{9}$$

Comparison of (7) with (8) shows that TM [4] in the near-Gaussian limit ($q \approx 1$) fully reproduces the log-normal result provided we take the following relation between μ and the non-extensivity parameter q:

$$\mu = \frac{9}{4\ln 2}(q-1)^2 \qquad q \approx 1. \tag{10}$$

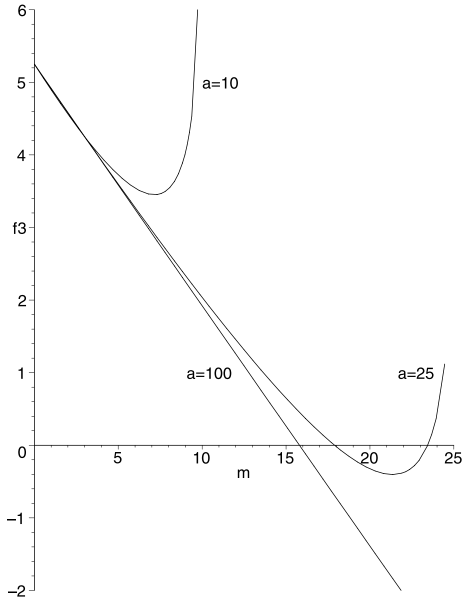

Figure 1. Plots of $f\ (= -\log_2(1-(m/a)) + \frac{1}{3}m\log_2(1-(3/a)) - \frac{1}{3}m - 3(\mu-2))$ against m for various values of the parameter $a = (3-q)/(q-1)$. Here $\mu = 0.25$.

4. Novikov inequality

If the moments of the local mean energy dissipation scale according to

$$\left\langle [\varepsilon(r)]^n \right\rangle \sim \langle \varepsilon \rangle^n \left(\frac{L}{r} \right)^{\mu_n} \tag{11}$$

then Novikov [5] showed that μ_n should increase at most linearly, i.e.

$$\mu_n \leqslant n + 3(\mu - 2) \qquad \text{for} \quad n > 2. \tag{12}$$

On recalling that

$$\left\langle |u|^m \right\rangle \sim \left\langle [\varepsilon(r)]^{m/3} \right\rangle r^{m/3} \tag{13}$$

and using (3), (4) and (10), the Novikov inequality (11) requires, for the Beck [4] formulation,

$$-\log_2 \left[\left\{ 1 - m \left(\frac{q-1}{3-q} \right) \right\} \left\{ 1 - 3 \left(\frac{q-1}{3-q} \right) \right\}^{-m/3} \right] \leqslant \frac{m}{3} + 3(\mu - 2). \tag{14}$$

In the near-Gaussian limit $|q-1| \ll 1$, equation (14) becomes

$$\frac{m(m-3)}{2\ln 2} \left(\frac{q-1}{3-q} \right)^2 \leqslant \frac{m}{3} + 3(m-2) \tag{15}$$

which is satisfied, if m is adequately large. However, for finite values of $(q-1)$, equation (14) is not satisfied, for any m, as demonstrated in figure 1.

5. Comparison with the multi-fractal formulation

According to the multi-fractal formulation of the velocity field in FDT (Meneveau and Sreenivasan [6]), the scaling exponent ζ_m of the mth-order structure function is given by

$$\zeta_m = \frac{m}{3} + \frac{1}{3}(3-m)\left(3 - D_{m/3}\right) \tag{16}$$

where D_n is the generalized fractal dimension of the energy dissipation field.

Comparing (16) with (4), we obtain the following relation between D_n and the non-extensivity parameter q:

$$D_{m/3} = 3\left\{1 - \left(\frac{1}{3-m}\right)\log_2\left[\left\{1 - m\left(\frac{q-1}{3-q}\right)\right\}\left\{1 - 3\left(\frac{q-1}{3-q}\right)\right\}^{-m/3}\right]\right\}$$
$$1 \leqslant q < 3. \qquad (17)$$

Equation (17), of course, yields in the limit $q \Rightarrow 1$,

$$D_{m/3} = 3 \qquad q \Rightarrow 1 \qquad (18)$$

as expected.

For $q \approx 1$, corresponding to the near-Gaussian limit, equation (17) gives the simple relation

$$D_{m/3} \approx 3\left[1 - \frac{m}{8\ln 2}(q-1)^2\right] \qquad q \approx 1. \qquad (19)$$

6. Comparison with the random-β model

In the random-β model (Benzi *et al* [7]), energy is transferred to only a fraction β of the eddies downstream in the cascade and the β's at the various levels of the cascade are allowed to vary randomly. In the modification given in Shivamoggi [8], in conformity with the direct numerical simulation results (Vincent and Meneguzzi [11] and others), filament-like structures (rather than sheet-like structures, as assumed in [7]) are assumed to be created with probability x ($0 \leqslant x \leqslant 1$), while space-filling eddies are created with probability $(1 - x)$. The scaling exponent ζ_m of the mth-order structure function is then given by [8],

$$\zeta_m = \frac{m}{3} - \log_2\left[x \cdot 4^{m/3-1} + (1-x)\right] \qquad 0 \leqslant x \leqslant 1. \qquad (20)$$

Comparing (20) with (4), we obtain a relation between x and the non-extensivity parameter q,

$$x = \frac{1}{4^{m/3-1} - 1}\left[\left\{1 - m\left(\frac{q-1}{3-q}\right)\right\}\left\{1 - 3\left(\frac{q-1}{3-q}\right)\right\}^{-m/3} - 1\right] \qquad 1 \leqslant q < 3. \qquad (21)$$

Equation (21) yields

$$m = 3 \qquad x = 0 \qquad (22)$$

as required.

On the other hand, equation (21) yields in the limit $q \Rightarrow 1$,

$$q \Rightarrow 1 \qquad x = 0 \qquad (23)$$

as expected.

For $q \approx 1$, corresponding to the near-Gaussian limit, equation (21) yields the simple relation

$$x \approx \frac{m(3-m)}{4^{m/3}}(q-1)^2 \qquad q \approx 1. \qquad (24)$$

7. Discussion

In this paper, we have made further investigations on the non-extensive statistical mechanics approach [4] to FDT. We have shown that this formulation is compatible with the multi-fractal formulation of FDT. In the near-Gaussian limit, this formulation is compatible with the log-normal model, as is to be desired. However, this formulation satisfies Novikov's inequality only in the near-Gaussian limit. We have established the relation between the non-extensivity parameter q and the parameters of other models.

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